A Model of Intra-seasonal Oscillations in the Earth atmosphere

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We suggest a way of rationalizing an intra-seasonal oscillations (IOs) of the Earth atmospheric flow as four meteorological relevant triads of interacting planetary waves, isolated from the system of all the rest planetary waves. Our model is independent of the topography (mountains, etc.) and gives a natural explanation of IOs both in the North and South Hemispheres. Spherical planetary waves are an example of a wave mesoscopic system obeying discrete resonances that also appears in other areas of physics.

Introduction. Concept of mesoscopic systems most often appears in condensed matter physics, e.g, in studying properties of superconductors on a scale comparable with that of the Cooper pairs[1], of miniaturized transistors on a computer chip, of disordered (glassy, granular) systems, when self-averaging is inefficient and fluctuations or the system prehistory become important. Similar situation occurs also in various natural phenomena – from wave turbulent systems in the Ocean [2] and Atmosphere, when wave lengthes are compatible with the Earth radius [3], to medicine [4], and even in sociology and economics, when finite size of a system (population, sociological group, market) becomes important [5]. Mesoscopic regimes are at the frontier between detailed, dynamical and statistical, self-averaging description of systems. Important observation for finite-size, weakly-decaying wave systems was made in [6]: discrete spacial-time resonances form small isolated clusters of interacting modes without energy exchange between the clusters. Clearly, there is exists relatively short, "threshold" wave, involving into the cluster with size large enough to "penetrate" into region of very short waves, where statistical description, that ignores the resonance discreetness, is valid.

Mathematical problem of finding these clusters in concrete cases is equivalent to solving some systems of high ($\gtrsim 12$) order Diophantine equations on a space of 6-8 variables in big integers [7]. Recently developed algorithms for their analysis [8] allows to find, in particular, all resonance clusters of atmospheric planetary waves, described by the spherical functions Y_ℓ^m with eigenvalues $|m| \leq \ell \leq 50$. It turned out that in this domain, consisting of 2500 spherical eigen-modes Y_ℓ^m, there exist only 20 different clusters involving only 103 different modes. Moreover, 15 of these clusters have the simplest "triad" structure, formed by three modes. Importantly, there are only four isolated triads in the domain $0 < m, \ell \leq 21$, which is meteorological significant for the problem of climate variability on intra-seasonal scale of about 10-100 days (waves with $\ell > 21$ have to short period to play significant role in this problem).

The main physical message of our Letter is that so-called *Intra-seasonal Oscillations (IOs) of the Earth atmospheric flow can be rationalized as periodical energy exchange within the above mentioned four isolated triads of the planetary waves*. IOs has been firstly have been detected [9] in the study of time series of tropical wind. Similar processes have been also discovered in the atmospheric angular momentum, atmospheric pressure, etc. Detailed analysis of the current state of the problem is presented in [10] and references therein; most part of the papers are devoted to the detection of these processes in some data sets, [11, 12] and to the reproducing them in computer simulations with comprehensive numerical models of the atmosphere [13]. Nevertheless, many aspects of the IOs remain unexplained, *e.g.* the reason of IOs in the North Hemisphere is supposed to be topography, see e.g. [14] and no reason is given for IOs in the South Hemisphere, there is no known way to predict the appearance of IOs, etc.

Our model considers IOs as intrinsic atmospheric phenomenon, related to a system of resonantly interacting triads of planetary waves, which is an example of wave mesoscopic system. The model is equally applied to the North and South Hemispheres, is independent (in the leading order) of the Earth topography, naturally have the period of desired order and allows to interpret the main observable features of IOs (see Sec. 5).

1. Atmospheric planetary waves. These waves are classically studied in the frame of barotropic vorticity equation on a sphere: [3]:

$$\partial \triangle \psi / \partial t + 2\, \partial \psi / \partial \lambda + J(\psi, \triangle \psi) = 0 \ . \qquad (1)$$

Here ψ is the dimensionless stream-function; velocity $\boldsymbol{v} = \Omega\, R\, [\mathbf{z} \times \boldsymbol{\nabla} \psi]$, with Ω being the angular velocity of the Earth and $\mathbf{z}$ – the vertical unit vector; the variables t, φ and λ denote dimensionless time (in the units of $1/\Omega$), latitude ($-\pi/2 \leq \varphi \leq \pi/2$) and longitude ($0 \leq \lambda \leq 2\pi$) respectively; $\triangle \psi$ and $J(a, b)$ are spherical Laplacian and Jacobian operators. The linear part of this equation has solutions in the form $A_j\, Y_{\ell_j}^{m_j}(\lambda, \varphi) \exp(i\omega_j t)$, $\omega_j = -2m_j/\ell_j(\ell_j+1)$. Integer parameters ℓ_j and $(\ell_j - m_j) \geq 0$ are longitudinal and latitudinal wave-numbers of j-mode, they are equal to the number of zeros of the spherical function along the longitude and latitude.

Assuming small level of nonlinearity, $|A_j| \ll 1$, we restrict ourselves by resonant interactions only. Under
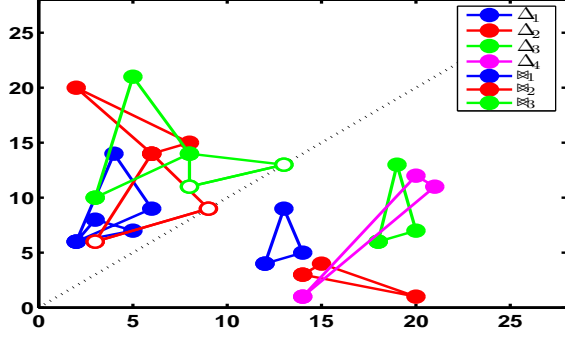
FIG. 1: Four isolated triads are shown below the diagonal dash line; ℓ – along horizontal axis and $m < \ell$ along the vertical one. The three clusters of two connected triads are shown above diagonal dash line (now ℓ – along the vertical axis and $m \leq \ell$ – along the horizontal one). The (meteorological significant) spectral domain is restricted by $0 < m, \ell \leq 21$. Localized modes are shown by full circles, delocalized ones (with relatively small interaction amplitude) by empty circles.

Triad	Modes $[m,\ell]$	Z	φ_0	$K[\mu]$	$10^6 E_0$	T_0
Δ_1	[4,12] [5,14] [9,13]	7.82	34	1.62	14.4	24
Δ_2	[3,14] [1,20] [4,15]	37.46	19	1.14	5.4	5
Δ_3	[6,18] [7,20] [13,19]	13.66	34	1.74	32.0	19
Δ_4	[1,14][11,21][12,20]	47.67	28	1.21	0.58	13
$\bowtie_1$	[2,6] [3,8] [5,7]	3.14	35	1.64	5.08	30
	[2,6] [4,14] [6,9]	14.63	37	1.61	0.395	10
$\bowtie_2$	[6,14] [2,20] [8,15]	69.25	31	1.13	0.61	8
	[3,6] [6,14] [9,9]	11.31	–	1.17	0.360	13
$\bowtie_3$	[3,10] [5,21] [8,14]	61.99	31	1.27	0.133	7
	[8,11] [5,21] [13,13]	8.71	–	1.36	0.784	24
$\boxtimes$	[1,6] [2,14] [3,9]	28.98	17	1.38	0.247	6
	[2,7] [11,20] [13,14]	2.77	42	1.08	1.78	26
	[1,6] [11,20] [12,15]	15.08	29	1.06	0.262	11
	[9,14] [3,20] [12,15]	74.93	50	1.36	0.487	8
	[3,9] [8,20] [11,14]	32.12	40	1.11	0.251	9
	[2,14][17,20][19,19]	11.05	–	1.05	3.33	24

Table 1. For each triad the following data are given: all resonantly interacting modes, interaction coefficient Z, interaction latitude φ_0 (in grad), magnitude of the elliptic integral $K(\mu)$, corresponding to the ESMRW December 1989 data for 500 hPa initial energy distribution in a triad, and initial dimensionless energy $E_0 * 10^6$ of each triad and resulting T_0 values (in days).

the resonance conditions for three modes: $\omega_1 + \omega_2 = \omega_3$, $m_1 + m_2 = m_3$, in which $|\ell_1 - \ell_2| < \ell_3 < \ell_1 + \ell_2$, and $\ell_1 + \ell_2 + \ell_3$ is odd, the triad amplitudes $A_j(t)$ varies in time according to equations [3]:

$$N_1 dA_1/dt = 2 Z N_{32} A_3 A_2^*, \quad N_2 dA_2/dt = 2 Z N_{13} A_1^* A_3,$$
$$N_3 dA_3/dt = 2 Z N_{21} A_1 A_2, \quad \text{for } |A_j| \ll 1 . \qquad (2)$$

Here $N_j \equiv \ell_j(\ell_j + 1)$, $N_{ij} \equiv N_i - N_j$, and interaction coefficient Z is an explicit function of wave numbers. This system conserves energy E and enstrophy H:

$$E = E_1 + E_2 + E_3, \quad H = N_1 E_1 + N_2 E_2 + N_3 E_3, \quad (3)$$

where the energy of j-mode is $E_j \equiv N_j |A_j|^2$.

2. Classification of the triads. Consider the structure and properties of interacting resonant triads in the meteorological significant domain $0 < m, \ell \leq 21$, where we found four isolated triads, denoted as $\Delta_1, \ldots, \Delta_4$, three "butterflies", i.e. clusters of two triads (denoted as $\bowtie_1$, $\bowtie_2$ and $\bowtie_3$) that are connected by a common mode, and one cluster of 6 connected triads denotes as $\boxtimes$. The structure of all isolated resonant triads and "butterflies" clusters is shown in Fig. 1. Main information about the triads in the chosen spectral domain is given in left 4 columns on Table 1: notations of the triads, three pair of ℓ_j, m_j for each triad, the value of the interaction coefficient Z and the so-called "interaction latitude" φ_0 introduced in Ref. [3]. Columns 5-7 contain data which is necessary to compute period of resonant interactions and will be commented on further.

We can interpret the latitude φ_0 as follows. The overlap of three wave-function in a triad, $\mathcal{Z}(\lambda, \varphi) \equiv Y_{\ell_1}^{m_1}(\lambda, \varphi) Y_{\ell_2}^{m_2}(\lambda, \varphi) Y_{\ell_2}^{m_3}(\lambda, \varphi)$ shows a contribution to the interaction coefficient $Z \propto \int \mathcal{Z}(\lambda, \varphi) \, d\lambda d\varphi$ from a particular location on the sphere. The overlap $\mathcal{Z}(\lambda, \varphi)$

has a maximum at a particular latitude φ_0 and a narrow latitudinal belt around φ_0 gives the main contribution to the global interaction amplitude Z. That is why φ_0 can be understood as the interaction latitude.

3. General solution of the triad equations. Linear change of variables $B_i = \alpha_i A_i$ with α_i being explicit functions on N_j allows to rewrite Sys. (2) as

$$\dot{B}_1 = 2 Z B_2^* B_3, \quad \dot{B}_2 = 2 Z B_1^* B_3, \quad \dot{B}_3 = 2 Z B_1 B_2 .$$

This system has two independent conservation laws

$$I_1 = |B_2|^2 + |B_3|^2 = (E N_1 - H) N_{23} / N_1 N_2 N_3, \quad (4a)$$
$$I_2 = |B_1|^2 + |B_3|^2 = (E N_2 - H) N_{13} / N_1 N_2 N_3 \quad (4b)$$

which are linear combinations of the energy E and enstrophy H. Direct calculations show that the general solution for B_i is expressed in Jacobian elliptic functions $B_1 = B_{1,0} \operatorname{cn}(\tau - \tau_0)$, $B_2 = B_{2,0} \operatorname{sn}(\tau - \tau_0)$, $B_3 = B_{3,0} \operatorname{dn}(\tau - \tau_0)$, where $B_{j,0}$, τ_0, are defined by initial conditions and $\tau = t/2 Z \sqrt[4]{I_1 I_2}$. Functions $\operatorname{cn}(\tau)$, $\operatorname{sn}(\tau)$ and $\operatorname{dn}(\tau)$ are periodic with the period $4K(\mu), 4K(\mu)$ and $2K(\mu)$ correspondingly, were $K(\mu)$

$$K(\mu) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \mu \sin^2 \theta}}, \quad \mu^2 \equiv \min\left\{\frac{I_1}{I_2}, \frac{I_2}{I_1}\right\} \leq 1.$$

Fig 2 illustrates the typical time dependence of all three dimensionless amplitudes of the triad Δ_1. One sees that

$K(\mu)$ is a smooth function that changes slowly enough such for the wide region of the initial conditions it can be roughly considered as a constant.

4. Period of triad oscillations. The period of energy exchange (measured in days) in the triads is given by $T = \pi\,K(\mu)\big/Z\sqrt[4]{I_1 I_2}$, that can be written a product of functions E and the *ratio of the enstrophy to the energy* $h \equiv H/E$:

$$T \quad\Rightarrow\quad T(E,h) = T_0(E)\,K(\mu)\,f(h)\,, \qquad (5a)$$

$$T_0(E) \;=\; \frac{\pi}{2\,Z\sqrt{2E}}\,\sqrt{\frac{N_1 N_2 N_3}{N_{21}\sqrt{N_{31}N_{23}}}}\,, \qquad (5b)$$

$$f(h) \;=\; \sqrt{N_{12}/2\sqrt{(N_1-h)(h-N_2)}}\,. \qquad (5c)$$

Here K depends on μ and, in its turn, μ depends on h as $\quad \mu^2(h) = \min\left[\,\dfrac{(h-N_2)N_{23}}{(N_1-h)N_{31}}\,,\dfrac{N_1-h)(N_{31})}{(h-N_2)N_{23}}\right].$ Equation (3) show that possible values of h lie inside one of the two intervals $N_2 \geq h \geq N_1$ or $N_1 \geq h \geq N_2$. Without loss of generality we set $N_2 \geq h \geq N_1$, then maximal possible value $h = N_2$ is realized if $E_1 = E_3 = 0$, i.e. only the second mode is excited. The minimal value $h = N_1$ is possible if $E_2 = E_3 = 0$, i.e. only the first mode is excited. In both cases, according to basic Eqs. (2) there is no time evolution, i.e. $E_j = $ const. for $j = 1, 2$ and 3. This is in agreement with Eq. (5c), according to which $f(h) \to \infty$ for $h \to N_2$ or $h \to N_1$.

Function $f(h)$, Eq. (5c), has a minimum (equal to one) just in the middle of the interval $N_2 \geq h \geq N_1$ at $h = h_+ \equiv (N_1 + N_2)/2$. For this value of h $\mu^2(h_+) = \min\left\{\dfrac{(N_2-N_3)}{(N_3-N_1)}\,,\dfrac{(N_3-N_1)}{(N_2-N_3)}\right\}$. For isolated triads of interest $\Delta_1\,,\Delta_2\,,\Delta_3$ and Δ_4, the values of $\mu^2(h_+)$ are 0.93, 0.41, 0.97 and 0.45 respectively, with $K(\mu)$ equal to 1.96, 1.27, 2.22 and 1.30.

The less trivial case of infinite period corresponds to $\mu = 1$, which is realized at $h = h_{\mathrm{crit}} \equiv N_1 + N_2 - N_3$. In this case $B_3(\tau) = \tanh\tau$ and for $\tau \to \infty$, $B_3 \to 1$ and B_1 and B_2 exponentially fast go to zero, i.e. for the specific value $h = h_{\mathrm{crit}}$, the high mode B_3 exponentially fast takes energy from two low modes. This is possible only for three particular values of h: $h = N_1$, N_2 and $N_1 + N_2 - N_3$. The h dependence of the period of the triads $\Delta_1\,, \dots\,, \Delta_4$, is presented in Fig. 2. One sees, that the regions, where the period exceeds twice the minimal possible are very narrow, just few percent of the available interval of h. This means, that though theoretically for *each triad we can always choose initial conditions in such a way that period will be large and even tend to infinity*, the probability of this is very small.

Indeed, for qualitative analysis we can think that in the turbulent atmosphere the probability to get some energy from global disturbances to a particular planetary wave is independent from the state of other waves and this probability is more or less the same for each wave in a triad. If so, the probability $\mathcal{P}(h)$ to have initial conditions with some value of h has to be a smooth function of h in the whole available interval $N_2 \geq h \geq N_1$. Roughly speaking, we can approximate $\mathcal{P}(h)$ as the constant: $\mathcal{P}(h) \simeq 1\big/(N_2 - N_1)$, $N_2 \geq h \geq N_1$. With this approximation we can, for example, for triads Δ_1–Δ_4 estimate the probability to have the period, twice exceeding the minimal one $T_0(E)$, Eq. (5b), as few percents. Moreover, as one sees in Fig. 2, the typical value of the period T is about $(1.4 \pm 0.1)T_0$ for the triads Δ_1, Δ_3 and about $(1.2 \pm 0.1)T_0$ for the triads Δ_2, Δ_4. This conclusion is in a qualitative agrement with the ECMWF (European Center for Medium-Range Weather Forcast) winter data, shown in Table 1, column 7.

5. Intra-seasonal Oscillations as Resonant Triads. Our interpretation of IOs as dynamical behavior of $\Delta_1\,, \Delta_2\,, \Delta_3$ and Δ_4 triads allows one to answer some questions appearing from meteorological observations [10].

• *What is the cause of IOs in South Hemisphere?* The basic fact of our model is the existence of global nonlinear interactions among planetary waves, independent of the topography.

• *Why the period of so-called "topographic" oscillation in North Hemisphere is given as 40 days by some researchers and 20-30 days - by other researchers?* The variations in the magnitudes of the period are caused by different initial energy and/or initial energy distribution among the modes of the same triad.

• *How do the tropical and mid-latitude oscillations interact?* Two mechanisms are possible: i) Triads with substantially different interaction latitudes belonging to the same group, for instance, triads [(1,6) (2,14) (3,9)] and [(3,9) (8,20) (11,14)] of ⊠ exchange their energies through other modes of this group and belong correspondingly to the tropical and extra-tropical latitudinal belts, and ii) Isolated triads can interact via some special modes called *active near-resonant modes* [6]. These modes have smallest resonance width with a given triad, and are themselves parts of some other resonant triad. For instance, the mode (13,19) is a near resonant for Δ_4 (with resonance discrepancy $\delta = 0.16$) and is resonant for Δ_3.

• *Why do the intra-seasonal oscillations are better observable in winter data?* In summer, modes have higher energies, periods of the triads become smaller, and resonances with big enough resonance width can destroy the clusters.

• *How to predict these recurrent features?* Amplitudes of the spherical harmonics with wave numbers taken from Table 1 have to be correlated: $\langle A_1(t)A_2(t)A_3^*(t)\rangle \sim \sqrt{\langle|A_1(t)|^2\rangle\langle|A_2(t)|^2\rangle\langle|A_3(t)|^2\rangle}$, see Fig. 2. Magnitudes of the expected periods can be computed beforehand by the given explicit formulae.

6. Conclusions

• Our simple model provides main robust features of IOs in terms of resonance clusters consisting of three
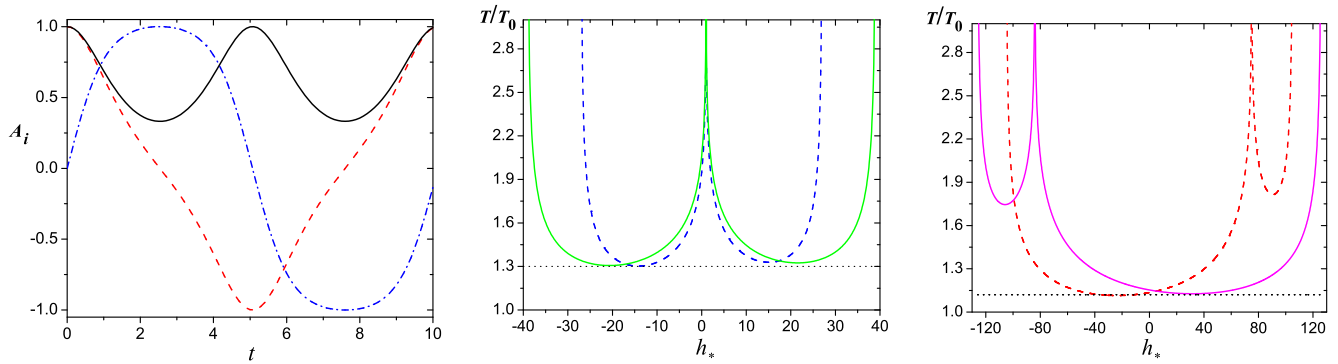
FIG. 2: Left panel: Time dependence of A_1, A_2 and A_3 [denoted by green solid, red dashed and blue dot-dashed lines accordingly] of the triad Δ_1, with $\mu = 0.89$, corresponding to observed data. Middle and Right panels: Dependence of the triad periods $T(E,h)/T_0(E)$, Eq. (5a). Middle panel: blue-dashed line corresponds to Δ_1 triad, green-solid line – Δ_3 triad. Right panel: Red dashed line – Δ_2 and magenta solid line – Δ_4 triad. Black dashed lines denote minimal period: $T \approx 1.3T_0$ for Δ_1, Δ_3 and $T \approx 1.18T_0$ for Δ_2, Δ_4.

modes of atmospheric waves.

• Energy behavior within the bigger clusters should be a subject of a special detailed study. Knowledge of cluster structure allows to simplify drastically their analysis. For instance, for "butterfly" cluster at least 6 real integrals of motions can be easily found. Universal method to construct isolated clusters and write out explicitly corresponding dynamical equations for a wide class of mesoscopic systems is given in [15].

• Our approach is quite general and can be used for studying many other mesoscopic systems, provided that explicit form of dispersion function $\varnothing(\boldsymbol{k})$ is known (here $\boldsymbol{k}$ is the wave vector of plane systems with periodical boundary conditions, or another set of eigenvalues in more complicated cases, like m, ℓ for the sphere). Properties of a specific mesoscopic system will depend on the 1) form of $\varnothing(\boldsymbol{k})$, 2) dimension of $\boldsymbol{k}$, 3) number of conservation laws, 4) initial magnitudes of the conserved values (energy, enstrophy, etc.) and their initial distribution among the modes in the cluster.

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